R.4 Factoring Polynomials

- Factoring Out the Greatest Common Factor
- Factoring by Grouping
- Factoring Trinomials
- Factoring Binomials
- Factoring by Substitution

The process of finding polynomials whose product equals a given polynomial is called **factoring.** Unless otherwise specified, we consider only integer coefficients when factoring polynomials. For example, since

$$4x + 12 = 4(x + 3)$$

both 4 and $x + 3$ are **factors** of $4x + 12$ and $4(x + 3)$ is a **factored form** of $4x + 12$.

A polynomial with variable terms that cannot be written as a product of two polynomials of lower degree is a **prime polynomial.** A polynomial is **factored completely** when it is written as a product of prime polynomials.

Factoring Out the Greatest Common Factor To factor $6x^2y^3 + 9xy^4 + 18y^5$, we look for a monomial that is the greatest common factor (GCF) of the three terms.

$$\begin{aligned} 6x^2y^3 + 9xy^4 + 18y^5 &= 3y^3(2x^2) + 3y^3(3xy) + 3y^3(6y^2) && \text{GCF} = 3y^3 \\ &= 3y^3(2x^2 + 3xy + 6y^2) && \text{Distributive property (Section R.2)} \end{aligned}$$

EXAMPLE 1 Factoring Out the Greatest Common Factor

Factor out the greatest common factor from each polynomial.

(a) $9y^5 + y^2$ **(b)** $6x^2t + 8xt + 12t$

(c) $14(m + 1)^3 - 28(m + 1)^2 - 7(m + 1)$

SOLUTION

(a) $9y^5 + y^2 = y^2(9y^3) + y^2(1)$ GCF $= y^2$

$= y^2(9y^3 + 1)$ Distributive property

Remember to include the 1.

CHECK Multiply out the factored form: $y^2(9y^3 + 1) = \overbrace{9y^5 + y^2}^{\text{Original polynomial}}$. ✓

(b) $6x^2t + 8xt + 12t = 2t(3x^2 + 4x + 6)$ GCF $= 2t$

CHECK $2t(3x^2 + 4x + 6) = 6x^2t + 8xt + 12t$ ✓

(c) $14(m + 1)^3 - 28(m + 1)^2 - 7(m + 1)$

$= 7(m + 1)[2(m + 1)^2 - 4(m + 1) - 1]$ GCF $= 7(m + 1)$

$= 7(m + 1)[2(m^2 + 2m + 1) - 4m - 4 - 1]$ Square $m + 1$ (Section R.3); distributive property

Remember the middle term.

$= 7(m + 1)(2m^2 + 4m + 2 - 4m - 4 - 1)$ Distributive property

$= 7(m + 1)(2m^2 - 3)$ Combine like terms.

✓ *Now Try Exercises 3, 9, and 15.*

CAUTION In **Example 1(a),** the 1 is essential in the answer, since

$$y^2(9y^3) \neq 9y^5 + y^2.$$

Factoring can always be checked by multiplying.

Factoring by Grouping When a polynomial has more than three terms, it can sometimes be factored using **factoring by grouping.** Consider this example.

$$
\begin{aligned}
ax + ay + 6x + 6y &= \underbrace{(ax + ay)}_{\substack{\text{Terms with}\\ \text{common}\\ \text{factor } a}} + \underbrace{(6x + 6y)}_{\substack{\text{Terms with}\\ \text{common}\\ \text{factor } 6}} && \text{Group the terms so that each group has a common factor.}\\
&= a(x + y) + 6(x + y) && \text{Factor each group.}\\
&= (x + y)(a + 6) && \text{Factor out } x + y.
\end{aligned}
$$

It is not always obvious which terms should be grouped. In cases like the one above, group in pairs. Experience and repeated trials are the most reliable tools.

EXAMPLE 2 Factoring by Grouping

Factor each polynomial by grouping.

(a) $mp^2 + 7m + 3p^2 + 21$ **(b)** $2y^2 + az - 2z - ay^2$

(c) $4x^3 + 2x^2 - 2x - 1$

SOLUTION

(a)
$$
\begin{aligned}
mp^2 + 7m + 3p^2 + 21 &= (mp^2 + 7m) + (3p^2 + 21) && \text{Group the terms.}\\
&= m(p^2 + 7) + 3(p^2 + 7) && \text{Factor each group.}\\
&= (p^2 + 7)(m + 3) && p^2 + 7 \text{ is a common factor.}
\end{aligned}
$$

CHECK
$$
\begin{aligned}
(p^2 + 7)(m + 3) &= mp^2 + 3p^2 + 7m + 21 && \text{FOIL (Section R.3)}\\
&= mp^2 + 7m + 3p^2 + 21 \checkmark && \text{Commutative property (Section R.2)}
\end{aligned}
$$

(b)
$$
\begin{aligned}
2y^2 + az - 2z - ay^2 &= 2y^2 - 2z - ay^2 + az && \text{Rearrange the terms.}\\
&= (2y^2 - 2z) + (-ay^2 + az) && \text{Group the terms.}\\
&= 2(y^2 - z) - a(y^2 - z) && \text{Factor out 2 and } -a \text{ so that } y^2 - z \text{ is a common factor.}\\
&= (y^2 - z)(2 - a) && \text{Factor out } y^2 - z.
\end{aligned}
$$

Be careful with signs here.

(c)
$$
\begin{aligned}
4x^3 + 2x^2 - 2x - 1 &= (4x^3 + 2x^2) + (-2x - 1) && \text{Group the terms.}\\
&= 2x^2(2x + 1) - 1(2x + 1) && \text{Factor each group.}\\
&= (2x + 1)(2x^2 - 1) && \text{Factor out } 2x + 1.
\end{aligned}
$$

✓ *Now Try Exercises 19 and 21.*

Factoring Trinomials As shown here, factoring is the opposite of multiplication.

Multiplication

$$(2x + 1)(3x - 4) = 6x^2 - 5x - 4$$

Factoring

One strategy in factoring trinomials requires using the FOIL method in reverse.

EXAMPLE 3 **Factoring Trinomials**

Factor each trinomial, if possible.

(a) $4y^2 - 11y + 6$

(b) $6p^2 - 7p - 5$

(c) $2x^2 + 13x - 18$

(d) $16y^3 + 24y^2 - 16y$

SOLUTION

(a) To factor this polynomial, we must find integers a, b, c, and d such that

$$4y^2 - 11y + 6 = (ay + b)(cy + d). \quad \text{FOIL}$$

Using FOIL, we see that $ac = 4$ and $bd = 6$. The positive factors of 4 are 4 and 1 or 2 and 2. Since the middle term has a negative coefficient, we consider only negative factors of 6. The possibilities are -2 and -3 or -1 and -6.

Now we try various arrangements of these factors until we find one that gives the correct coefficient of y.

$$(2y - 1)(2y - 6) = 4y^2 - 14y + 6 \quad \text{Incorrect}$$
$$(2y - 2)(2y - 3) = 4y^2 - 10y + 6 \quad \text{Incorrect}$$
$$(y - 2)(4y - 3) = 4y^2 - 11y + 6 \quad \text{Correct}$$

Therefore, $4y^2 - 11y + 6$ factors as $(y - 2)(4y - 3)$.

CHECK $(y - 2)(4y - 3) = 4y^2 - 3y - 8y + 6$ FOIL

$= 4y^2 - 11y + 6$ ✓ Original polynomial

(b) Again, we try various possibilities to factor $6p^2 - 7p - 5$. The positive factors of 6 could be 2 and 3 or 1 and 6. As factors of -5 we have only -1 and 5 or -5 and 1.

$$(2p - 5)(3p + 1) = 6p^2 - 13p - 5 \quad \text{Incorrect}$$
$$(3p - 5)(2p + 1) = 6p^2 - 7p - 5 \quad \text{Correct}$$

Thus, $6p^2 - 7p - 5$ factors as $(3p - 5)(2p + 1)$.

(c) If we try to factor $2x^2 + 13x - 18$ as above, we find that none of the pairs of factors gives the correct coefficient of x.

$$(2x + 9)(x - 2) = 2x^2 + 5x - 18 \quad \text{Incorrect}$$
$$(2x - 3)(x + 6) = 2x^2 + 9x - 18 \quad \text{Incorrect}$$
$$(2x - 1)(x + 18) = 2x^2 + 35x - 18 \quad \text{Incorrect}$$

Additional trials are also unsuccessful. Thus, this trinomial cannot be factored with integer coefficients and is prime.

(d) $16y^3 + 24y^2 - 16y = 8y(2y^2 + 3y - 2)$ Factor out the GCF, $8y$.

$= 8y(2y - 1)(y + 2)$ Factor the trinomial.

Remember to include the common factor in the final form.

✓ *Now Try Exercises 25, 27, 29, and 31.*

NOTE In **Example 3,** we chose positive factors of the positive first term. We could have used two negative factors, but the work is easier if positive factors are used.

Each of the special patterns for multiplication given in **Section R.3** can be used in reverse to get a pattern for factoring. Perfect square trinomials can be factored as follows.

Factoring Perfect Square Trinomials

$$x^2 + 2xy + y^2 = (x + y)^2$$
$$x^2 - 2xy + y^2 = (x - y)^2$$

EXAMPLE 4 Factoring Perfect Square Trinomials

Factor each trinomial.

(a) $16p^2 - 40pq + 25q^2$ **(b)** $36x^2y^2 + 84xy + 49$

SOLUTION

(a) Since $16p^2 = (4p)^2$ and $25q^2 = (5q)^2$, we use the second pattern shown in the box with $4p$ replacing x and $5q$ replacing y.

$$16p^2 - 40pq + 25q^2 = (4p)^2 - 2(4p)(5q) + (5q)^2$$
$$= (4p - 5q)^2$$

Make sure that the middle term of the trinomial being factored, $-40pq$ here, is twice the product of the two terms in the binomial $4p - 5q$.

$$-40pq = 2(4p)(-5q)$$

Thus, $16p^2 - 40pq + 25q^2$ factors as $(4p - 5q)^2$.

CHECK $(4p - 5q)^2 = 16p^2 - 40pq + 25q^2$ ✓ Multiply.

(b) $36x^2y^2 + 84xy + 49$ factors as $(6xy + 7)^2$ $2(6xy)(7) = 84xy$

CHECK Square $6xy + 7$: $(6xy + 7)^2 = 36x^2y^2 + 84xy + 49$. ✓

✓ *Now Try Exercises 41 and 45.*

Factoring Binomials Check first to see whether the terms of a binomial have a common factor. If so, factor it out. The binomial may also fit one of the following patterns.

Factoring Binomials

Difference of Squares	$x^2 - y^2 = (x + y)(x - y)$
Difference of Cubes	$x^3 - y^3 = (x - y)(x^2 + xy + y^2)$
Sum of Cubes	$x^3 + y^3 = (x + y)(x^2 - xy + y^2)$

CAUTION ***There is no factoring pattern for a sum of squares in the real number system.*** In particular, $x^2 + y^2$ **does not factor as** $(x + y)^2$, for real numbers x and y.

EXAMPLE 5 **Factoring Differences of Squares**

Factor each polynomial.

(a) $4m^2 - 9$ **(b)** $256k^4 - 625m^4$ **(c)** $(a + 2b)^2 - 4c^2$

(d) $x^2 - 6x + 9 - y^4$ **(e)** $y^2 - x^2 + 6x - 9$

SOLUTION

(a) $4m^2 - 9 = (2m)^2 - 3^2$ Write as a difference of squares.

$= (2m + 3)(2m - 3)$ Factor.

Check by multiplying.

(b) $256k^4 - 625m^4 = (16k^2)^2 - (25m^2)^2$ Write as a difference of squares.

Don't stop here. $= (16k^2 + 25m^2)(16k^2 - 25m^2)$ Factor.

$= (16k^2 + 25m^2)(4k + 5m)(4k - 5m)$ Factor $16k^2 - 25m^2$.

CHECK $(16k^2 + 25m^2)(4k + 5m)(4k - 5m)$

$= (16k^2 + 25m^2)(16k^2 - 25m^2)$ Multiply the last two factors.

$= 256k^4 - 625m^4$ ✓ Original polynomial

(c) $(a + 2b)^2 - 4c^2 = (a + 2b)^2 - (2c)^2$ Write as a difference of squares.

$= [(a + 2b) + 2c][(a + 2b) - 2c]$ Factor.

$= (a + 2b + 2c)(a + 2b - 2c)$

Check by multiplying.

(d) $x^2 - 6x + 9 - y^4 = (x^2 - 6x + 9) - y^4$ Group terms.

$= (x - 3)^2 - y^4$ Factor the trinomial.

$= (x - 3)^2 - (y^2)^2$ Write as a difference of squares.

$= [(x - 3) + y^2][(x - 3) - y^2]$ Factor.

$= (x - 3 + y^2)(x - 3 - y^2)$

Check by multiplying.

(e) $y^2 - x^2 + 6x - 9 = y^2 - (x^2 - 6x + 9)$ Factor out the negative sign and group the last three terms.

Be careful with signs. This is a perfect square trinomial.

$= y^2 - (x - 3)^2$ Write as a difference of squares.

$= [y - (x - 3)][y + (x - 3)]$ Factor.

$= (y - x + 3)(y + x - 3)$ Distributive property

Check by multiplying.

✓ *Now Try Exercises 51, 55, 57, and 59.*

CAUTION When factoring as in **Example 5(e),** be careful with signs. Inserting an open parenthesis following the minus sign requires changing the signs of all of the following terms.

EXAMPLE 6 Factoring Sums or Differences of Cubes

Factor each polynomial.

(a) $x^3 + 27$ **(b)** $m^3 - 64n^3$ **(c)** $8q^6 + 125p^9$

SOLUTION

(a) $x^3 + 27 = x^3 + 3^3$ Write as a sum of cubes.

$= (x + 3)(x^2 - 3x + 3^2)$ Factor.

$= (x + 3)(x^2 - 3x + 9)$ Apply the exponent.

(b) $m^3 - 64n^3 = m^3 - (4n)^3$ Write as a difference of cubes.

$= (m - 4n)[m^2 + m(4n) + (4n)^2]$ Factor.

$= (m - 4n)(m^2 + 4mn + 16n^2)$ Simplify.

(c) $8q^6 + 125p^9 = (2q^2)^3 + (5p^3)^3$ Write as a sum of cubes.

$= (2q^2 + 5p^3)[(2q^2)^2 - 2q^2(5p^3) + (5p^3)^2]$ Factor.

$= (2q^2 + 5p^3)(4q^4 - 10q^2p^3 + 25p^6)$ Simplify.

✔ *Now Try Exercises 61, 63, and 65.*

Factoring by Substitution We introduce a new technique for factoring.

EXAMPLE 7 Factoring by Substitution

Factor each polynomial.

(a) $10(2a - 1)^2 - 19(2a - 1) - 15$ **(b)** $(2a - 1)^3 + 8$

(c) $6z^4 - 13z^2 - 5$

SOLUTION

(a) $10(2a - 1)^2 - 19(2a - 1) - 15$ Replace $2a - 1$ with u so that $(2a - 1)^2$ becomes u^2.

$= 10u^2 - 19u - 15$

$= (5u + 3)(2u - 5)$ Factor.

Don't stop here. Replace u with $2a - 1$.

$= [5(2a - 1) + 3][2(2a - 1) - 5]$ Replace u with $2a - 1$.

$= (10a - 5 + 3)(4a - 2 - 5)$ Distributive property

$= (10a - 2)(4a - 7)$ Simplify.

$= 2(5a - 1)(4a - 7)$ Factor out the common factor.

(b) $(2a - 1)^3 + 8 = u^3 + 8$ Replace $2a - 1$ with u.

$= u^3 + 2^3$ Write as a sum of cubes.

$= (u + 2)(u^2 - 2u + 4)$ Factor.

$= [(2a - 1) + 2][(2a - 1)^2 - 2(2a - 1) + 4]$
Replace u with $2a - 1$.

$= (2a + 1)(4a^2 - 4a + 1 - 4a + 2 + 4)$
Add, and then multiply.

$= (2a + 1)(4a^2 - 8a + 7)$ Combine like terms.

(c) $6z^4 - 13z^2 - 5 = 6u^2 - 13u - 5$ Replace z^2 with u.

Remember to make the final substitution.

$= (2u - 5)(3u + 1)$ Use FOIL to factor.

$= (2z^2 - 5)(3z^2 + 1)$ Replace u with z^2.

✔ *Now Try Exercises 79 and 83.*