R.2 Real Numbers and Their Properties

- Sets of Numbers and the Number Line
- Exponents
- Order of Operations
- Properties of Real Numbers
- Order on the Number Line
- Absolute Value

Sets of Numbers and the Number Line As mentioned in the previous section, the set of **natural numbers** is written in set notation as follows.

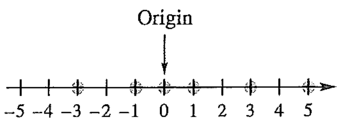
$$\{1, 2, 3, 4, \ldots\} \quad \text{Natural numbers (Section R.1)}$$

Including 0 with the set of natural numbers gives the set of **whole numbers.**

$$\{0, 1, 2, 3, 4, \ldots\} \quad \text{Whole numbers}$$

Including the negatives of the natural numbers with the set of whole numbers gives the set of **integers.**

$$\{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\} \quad \text{Integers}$$



Graph of the Set $\{-3, -1, 0, 1, 3, 5\}$

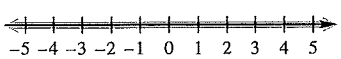
Figure 5

Integers can be **graphed** on a **number line.** See **Figure 5**. Every number corresponds to one and only one point on the number line, and each point corresponds to one and only one number. The number associated with a given point is called the **coordinate** of the point. This correspondence forms a **coordinate system.**

The result of dividing two integers (with a nonzero divisor) is called a *rational number,* or *fraction.* A **rational number** is an element of the set defined as follows.

$$\left\{ \frac{p}{q} \,\middle|\, p \text{ and } q \text{ are integers and } q \neq 0 \right\} \quad \text{Rational numbers}$$

The set of rational numbers includes the natural numbers, the whole numbers, and the integers. For example, the integer -3 is a rational number because it can be written as $\frac{-3}{1}$. Numbers that can be written as repeating or terminating decimals are also rational numbers. For example, $0.\overline{6} = 0.66666\ldots$ represents a rational number that can be expressed as the fraction $\frac{2}{3}$.

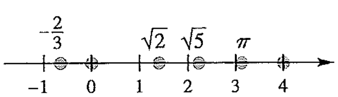


Graph of the Set of Real Numbers

Figure 6

The set of all numbers that correspond to points on a number line is the **real numbers,** shown in **Figure 6**. Real numbers can be represented by decimals. Since every fraction has a decimal form—for example, $\frac{1}{4} = 0.25$—real numbers include rational numbers.

Some real numbers cannot be represented by quotients of integers. These numbers are **irrational numbers.** The set of irrational numbers includes $\sqrt{3}$ and $\sqrt{5}$. Another irrational number is π, which is *approximately* equal to 3.14159. The numbers in the set $\left\{-\frac{2}{3}, 0, \sqrt{2}, \sqrt{5}, \pi, 4\right\}$ can be located on a number line, as shown in **Figure 7**.



$\sqrt{2}$, $\sqrt{5}$, and π are irrational. Since $\sqrt{2}$ is approximately equal to 1.41, it is located between 1 and 2, slightly closer to 1.

Figure 7

The sets of numbers discussed so far are summarized as follows.

Sets of Numbers

Set	Description
Natural numbers	$\{1, 2, 3, 4, \ldots\}$
Whole numbers	$\{0, 1, 2, 3, 4, \ldots\}$
Integers	$\{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}$
Rational numbers	$\left\{\frac{p}{q} \mid p \text{ and } q \text{ are integers and } q \neq 0\right\}$
Irrational numbers	$\{x \mid x \text{ is real but not rational}\}$
Real numbers	$\{x \mid x \text{ corresponds to a point on a number line}\}$

EXAMPLE 1 **Identifying Sets of Numbers**

Let $A = \left\{-8, -6, -\frac{12}{4}, -\frac{3}{4}, 0, \frac{3}{8}, \frac{1}{2}, 1, \sqrt{2}, \sqrt{5}, 6\right\}$. List the elements from A that belong to each set.

(a) Natural numbers **(b)** Whole numbers **(c)** Integers

(d) Rational numbers **(e)** Irrational numbers **(f)** Real numbers

SOLUTION

(a) Natural numbers: 1 and 6 **(b)** Whole numbers: 0, 1, and 6

(c) Integers: -8, -6, $-\frac{12}{4}$ (or -3), 0, 1, and 6

(d) Rational numbers: -8, -6, $-\frac{12}{4}$ (or -3), $-\frac{3}{4}$, 0, $\frac{3}{8}$, $\frac{1}{2}$, 1, and 6

(e) Irrational numbers: $\sqrt{2}$ and $\sqrt{5}$

(f) All elements of A are real numbers.

Now Try Exercises 1, 11, and 13.

Figure 8 shows the relationships among the subsets of the real numbers.

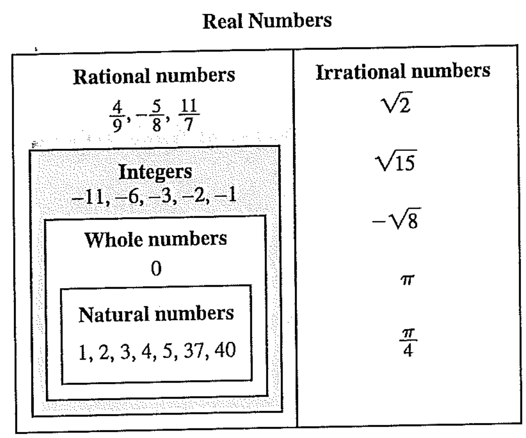


Figure 8

Exponents The product $2 \cdot 2 \cdot 2$ can be written as 2^3, where the 3 shows that three factors of 2 appear in the product.

Exponential Notation

If n is any positive integer and a is any real number, then the nth power of a is written using exponential notation as follows.

$$a^n = \underbrace{a \cdot a \cdot a \cdot \ldots \cdot a}_{n \text{ factors of } a}$$

That is, a^n means the product of n factors of a. The integer n is the **exponent,** a is the **base,** and a^n is a **power** or an **exponential expression** (or simply an **exponential**). Read a^n as **"a to the nth power,"** or just **"a to the nth."**

EXAMPLE 2 **Evaluating Exponential Expressions**

Evaluate each exponential expression, and identify the base and the exponent.

(a) 4^3 **(b)** $(-6)^2$ **(c)** -6^2 **(d)** $4 \cdot 3^2$ **(e)** $(4 \cdot 3)^2$

SOLUTION

(a) $4^3 = \underbrace{4 \cdot 4 \cdot 4}_{\text{3 factors of 4}} = 64$ The base is 4 and the exponent is 3.

(b) $(-6)^2 = (-6)(-6) = 36$ The base is -6 and the exponent is 2.

(c) $-6^2 = -(6 \cdot 6) = -36$ Notice that parts (b) and (c) are different.
The base is 6 and the exponent is 2.

(d) $4 \cdot 3^2 = 4 \cdot 3 \cdot 3 = 36$ The base is 3 and the exponent is 2.
$3^2 = 3 \cdot 3$, NOT $3 \cdot 2$

(e) $(4 \cdot 3)^2 = 12^2 = 144$ $(4 \cdot 3)^2 \neq 4 \cdot 3^2$
The base is $4 \cdot 3$, or 12, and the exponent is 2.

Now Try Exercises 17, 19, 21, and 23.

Order of Operations When a problem involves more than one operation symbol, we use the following order of operations.

Order of Operations

If grouping symbols such as parentheses, square brackets, absolute value bars, or fraction bars are present, begin as follows.

Step 1 Work separately above and below each **fraction bar.**

Step 2 Use the rules below within each set of **parentheses** or **square brackets.** Start with the innermost set and work outward.

If no grouping symbols are present, follow these steps.

Step 1 Simplify all **powers** and **roots.** ***Work from left to right.***

Step 2 Do any **multiplications** or **divisions** in order. ***Work from left to right.***

Step 3 Do any **negations, additions,** or **subtractions** in order. ***Work from left to right.***

EXAMPLE 3 **Using Order of Operations**

Evaluate each expression.

(a) $6 \div 3 + 2^3 \cdot 5$ **(b)** $(8 + 6) \div 7 \cdot 3 - 6$

(c) $\dfrac{4 + 3^2}{6 - 5 \cdot 3}$ **(d)** $\dfrac{-(-3)^3 + (-5)}{2(-8) - 5(3)}$

SOLUTION

(a)
$$\begin{aligned} 6 \div 3 + 2^3 \cdot 5 &= 6 \div 3 + 8 \cdot 5 && \text{Evaluate the exponential.} \\ &= 2 + 8 \cdot 5 && \text{Divide.} \\ &= 2 + 40 && \text{Multiply.} \\ &= 42 && \text{Add.} \end{aligned}$$

Multiply or divide *in order from left to right.*

(b) $(8 + 6) \div 7 \cdot 3 - 6 = 14 \div 7 \cdot 3 - 6$ Work inside parentheses.

Be careful to divide *before* multiplying here.

$= 2 \cdot 3 - 6$ Divide.

$= 6 - 6$ Multiply.

$= 0$ Subtract.

(c) $\dfrac{4 + 3^2}{6 - 5 \cdot 3} = \dfrac{4 + 9}{6 - 15}$ Evaluate the exponential and multiply.

$= \dfrac{13}{-9}$, or $-\dfrac{13}{9}$ Add and subtract; $\frac{a}{-b} = -\frac{a}{b}$.

(d) $\dfrac{-(-3)^3 + (-5)}{2(-8) - 5(3)} = \dfrac{-(-27) + (-5)}{2(-8) - 5(3)}$ Evaluate the exponential.

$= \dfrac{27 + (-5)}{-16 - 15}$ Multiply.

$= \dfrac{22}{-31}$, or $-\dfrac{22}{31}$ Add and subtract; $\frac{a}{-b} = -\frac{a}{b}$.

Now Try Exercises 25, 27, and 33.

EXAMPLE 4 Using Order of Operations

Evaluate each expression for $x = -2$, $y = 5$, and $z = -3$.

(a) $-4x^2 - 7y + 4z$ **(b)** $\dfrac{2(x-5)^2 + 4y}{z + 4}$ **(c)** $\dfrac{\frac{x}{2} - \frac{y}{5}}{\frac{3z}{9} + \frac{8y}{5}}$

SOLUTION

Use parentheses around substituted values to avoid errors.

(a) $-4x^2 - 7y + 4z = -4(-2)^2 - 7(5) + 4(-3)$ Substitute: $x = -2$, $y = 5$, and $z = -3$.

$= -4(4) - 7(5) + 4(-3)$ Evaluate the exponential.

$= -16 - 35 - 12$ Multiply.

$= -63$ Subtract.

(b) $\dfrac{2(x-5)^2 + 4y}{z + 4} = \dfrac{2(-2-5)^2 + 4(5)}{-3 + 4}$ Substitute: $x = -2$, $y = 5$, and $z = -3$.

$= \dfrac{2(-7)^2 + 20}{1}$ Work inside parentheses. Then multiply and add.

$= 2(49) + 20$ Evaluate the exponential.

$= 98 + 20$ Multiply.

$= 118$ Add.

(c) $\dfrac{\frac{x}{2} - \frac{y}{5}}{\frac{3z}{9} + \frac{8y}{5}} = \dfrac{\frac{-2}{2} - \frac{5}{5}}{\frac{3(-3)}{9} + \frac{8(5)}{5}}$ Substitute: $x = -2$, $y = 5$, and $z = -3$.

$= \dfrac{-1 - 1}{-1 + 8}$, or $-\dfrac{2}{7}$ Simplify the fractions.

Now Try Exercises 35, 43, and 45.

Properties of Real Numbers The following basic properties can be generalized to apply to expressions with variables.

Properties of Real Numbers

Let a, b, and c represent real numbers.

Property	Description
Closure Properties $a + b$ is a real number. ab is a real number.	The sum or product of two real numbers is a real number.
Commutative Properties $a + b = b + a$ $ab = ba$	The sum or product of two real numbers is the same regardless of their order.
Associative Properties $(a + b) + c = a + (b + c)$ $(ab)c = a(bc)$	The sum or product of three real numbers is the same no matter which two are added or multiplied first.
Identity Properties There exists a unique real number 0 such that $a + 0 = a$ and $0 + a = a.$ There exists a unique real number 1 such that $a \cdot 1 = a$ and $1 \cdot a = a.$	The sum of a real number and 0 is that real number, and the product of a real number and 1 is that real number.
Inverse Properties There exists a unique real number $-a$ such that $a + (-a) = 0$ and $-a + a = 0.$ If $a \neq 0$, there exists a unique real number $\frac{1}{a}$ such that $a \cdot \frac{1}{a} = 1$ and $\frac{1}{a} \cdot a = 1.$	The sum of any real number and its negative is 0, and the product of any nonzero real number and its reciprocal is 1.
Distributive Properties $a(b + c) = ab + ac$ $a(b - c) = ab - ac$	The product of a real number and the sum (or difference) of two real numbers equals the sum (or difference) of the products of the first number and each of the other numbers.

The **multiplication property of zero** says that $0 \cdot a = a \cdot 0 = 0$ for all real numbers a.

CAUTION With the commutative properties, the *order* changes, but with the associative properties, the *grouping* changes.

Commutative Properties	Associative Properties
$(x + 4) + 9 = (4 + x) + 9$	$(x + 4) + 9 = x + (4 + 9)$
$7 \cdot (5 \cdot 2) = (5 \cdot 2) \cdot 7$	$7 \cdot (5 \cdot 2) = (7 \cdot 5) \cdot 2$

EXAMPLE 5 Simplifying Expressions

Use the commutative and associative properties to simplify each expression.

(a) $6 + (9 + x)$ **(b)** $\frac{5}{8}(16y)$ **(c)** $-10p\left(\frac{6}{5}\right)$

SOLUTION

(a) $6 + (9 + x) = (6 + 9) + x$ Associative property

$= 15 + x$ Add.

(b) $\frac{5}{8}(16y) = \left(\frac{5}{8} \cdot 16\right)y$ Associative property

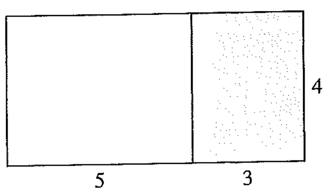
$= 10y$ Multiply.

(c) $-10p\left(\frac{6}{5}\right) = \frac{6}{5}(-10p)$ Commutative property

$= \left[\frac{6}{5}(-10)\right]p$ Associative property

$= -12p$ Multiply.

Now Try Exercises 63 and 65.



Geometric Model of the Distributive Property

Figure 9

Figure 9 helps to explain the distributive property. The area of the entire region shown can be found in two ways, as follows.

$$4(5 + 3) = 4(8) = 32$$

or

$$4(5) + 4(3) = 20 + 12 = 32$$

The result is the same. This means that

$$4(5 + 3) = 4(5) + 4(3).$$

EXAMPLE 6 Using the Distributive Property

Rewrite each expression using the distributive property and simplify, if possible.

(a) $3(x + y)$ **(b)** $-(m - 4n)$ **(c)** $\frac{1}{3}\left(\frac{4}{5}m - \frac{3}{2}n - 27\right)$ **(d)** $7p + 21$

SOLUTION

(a) $3(x + y) = 3x + 3y$

(b) $-(m - 4n) = -1(m - 4n)$

$= -1(m) + (-1)(-4n)$

$= -m + 4n$

Be careful with the negative signs.

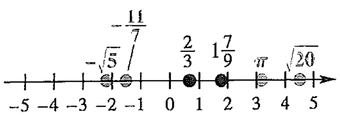
(c) $\frac{1}{3}\left(\frac{4}{5}m - \frac{3}{2}n - 27\right) = \frac{1}{3}\left(\frac{4}{5}m\right) + \frac{1}{3}\left(-\frac{3}{2}n\right) + \frac{1}{3}(-27)$

$= \frac{4}{15}m - \frac{1}{2}n - 9$

(d) $7p + 21 = 7p + 7 \cdot 3$

$= 7(p + 3)$ Distributive property in reverse

Now Try Exercises 67, 69, and 71.



$-\sqrt{5}$ is to the left of $-\frac{11}{7}$ on the number line, so $-\sqrt{5} < -\frac{11}{7}$, and $\sqrt{20}$ is to the right of π, indicating that $\sqrt{20} > \pi$.

Figure 10

Order on the Number Line If the real number a is to the left of the real number b on a number line, then

$$\textit{a is less than b,} \quad \text{written} \quad a < b.$$

If a is to the right of b, then

$$\textit{a is greater than b,} \quad \text{written} \quad a > b.$$

The inequality symbol must point toward the lesser number.

Figure 10 illustrates this with several pairs of numbers. Statements involving these symbols, as well as the symbols less than or equal to, $\leq$, and greater than or equal to, $\geq$, are called **inequalities.** The inequality $a < b < c$ says that b is ***between*** a and c since $a < b$ and $b < c$.

Absolute Value The distance on the number line from a number to 0 is called the **absolute value** of that number. The absolute value of the number a is written $|a|$. For example, the distance on the number line from 5 to 0 is 5, as is the distance from -5 to 0. See **Figure 11.** Therefore, both of the following are true.

$$|5| = 5 \quad \text{and} \quad |-5| = 5$$

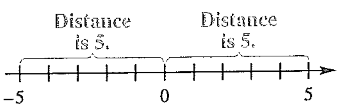


Figure 11

NOTE ***Since distance cannot be negative, the absolute value of a number is always positive or 0.***

The algebraic definition of absolute value follows.

Absolute Value

Let a represent a real number.

$$|a| = \begin{cases} a & \text{if } a \geq 0 \\ -a & \text{if } a < 0 \end{cases}$$

That is, the absolute value of a positive number or 0 equals that number, while the absolute value of a negative number equals its negative (or opposite).

EXAMPLE 7 Evaluating Absolute Values

Evaluate each expression.

(a) $\left|-\frac{5}{8}\right|$ **(b)** $-|8|$ **(c)** $-|-2|$ **(d)** $|2x|$, for $x = \pi$

SOLUTION

(a) $\left|-\frac{5}{8}\right| = \frac{5}{8}$

(b) $-|8| = -(8) = -8$

(c) $-|-2| = -(2) = -2$

(d) $|2\pi| = 2\pi$

Now Try Exercises 83 and 87.

Absolute value is useful in applications where only the *size* (or magnitude), not the *sign,* of the difference between two numbers is important.

EXAMPLE 8 **Measuring Blood Pressure Difference**

Systolic blood pressure is the maximum pressure produced by each heartbeat. Both low blood pressure and high blood pressure may be cause for medical concern. Therefore, health care professionals are interested in a patient's "pressure difference from normal," or P_d.

If 120 is considered a normal systolic pressure, then

$P_d = |P - 120|$, where P is the patient's recorded systolic pressure.

Find P_d for a patient with a systolic pressure, P, of 113.

SOLUTION

$$
\begin{aligned}
P_d &= |P - 120| \\
&= |113 - 120| && \text{Let } P = 113. \\
&= |-7| && \text{Subtract.} \\
&= 7 && \text{Definition of absolute value}
\end{aligned}
$$

Now Try Exercise 89.

Properties of Absolute Value

Let a and b represent real numbers.

Property	Description
1. $\|a\| \geq 0$	The absolute value of a real number is positive or 0.
2. $\|-a\| = \|a\|$	The absolute values of a real number and its opposite are equal.
3. $\|a\| \cdot \|b\| = \|ab\|$	The product of the absolute values of two real numbers equals the absolute value of their product.
4. $\dfrac{\|a\|}{\|b\|} = \left\|\dfrac{a}{b}\right\|$ $(b \neq 0)$	The quotient of the absolute values of two real numbers equals the absolute value of their quotient.
5. $\|a + b\| \leq \|a\| + \|b\|$ (the triangle inequality)	The absolute value of the sum of two real numbers is less than or equal to the sum of their absolute values.

LOOKING AHEAD TO CALCULUS

One of the most important definitions in calculus, that of the **limit,** uses absolute value. (The symbols ϵ (epsilon) and δ (delta) are often used to represent small quantities in mathematics.)

Suppose that a function f is defined at every number in an open interval I containing a, except perhaps at a itself. Then the limit of $f(x)$ as x approaches a is L, written

$$\lim_{x \to a} f(x) = L,$$

if for every $\epsilon > 0$ there exists a $\delta > 0$ such that $|f(x) - L| < \epsilon$ whenever $0 < |x - a| < \delta$.

To illustrate these properties, see the following.

$|-15| = 15$ and $15 \geq 0$ — Property 1

$|-10| = 10$ and $|10| = 10$, so $|-10| = |10|$. — Property 2

$|5x| = |5| \cdot |x| = 5|x|$ since 5 is positive. — Property 3

$\left|\dfrac{2}{y}\right| = \dfrac{|2|}{|y|} = \dfrac{2}{|y|}$, $y \neq 0$ — Property 4

To illustrate the triangle inequality, we let $a = 3$ and $b = -7$.

$$|a + b| = |3 + (-7)| = |-4| = 4$$

$$|a| + |b| = |3| + |-7| = 3 + 7 = 10$$

Thus, $|a + b| \leq |a| + |b|$. — Property 5

NOTE As seen in **Example 9(b),** absolute value bars can also act as symbols of inclusion. Remember this when applying the rules for order of operations.

EXAMPLE 9 Evaluating Absolute Value Expressions

Let $x = -6$ and $y = 10$. Evaluate each expression.

(a) $|2x - 3y|$ **(b)** $\dfrac{2|x| - |3y|}{|xy|}$

SOLUTION

(a) $|2x - 3y| = |2(-6) - 3(10)|$ Substitute.

$= |-12 - 30|$ Work inside absolute value bars. Multiply.

$= |-42|$ Subtract.

$= 42$ Definition of absolute value

(b) $\dfrac{2|x| - |3y|}{|xy|} = \dfrac{2|-6| - |3(10)|}{|-6(10)|}$ Substitute.

$= \dfrac{2 \cdot 6 - |30|}{|-60|}$ $|-6| = 6$; multiply.

$= \dfrac{12 - 30}{60}$ Multiply. $|30| = 30$, $|-60| = 60$

$= \dfrac{-18}{60}$ Subtract.

$= -\dfrac{3}{10}$ Write in lowest terms; $\frac{-a}{b} = -\frac{a}{b}$.

Now Try Exercises 93 and 95.

Distance between Points on a Number Line

If P and Q are points on a number line with coordinates a and b, respectively, then the distance $d(P, Q)$ between them is given by the following.

$$d(P, Q) = |b - a| \quad \text{or} \quad d(P, Q) = |a - b|$$

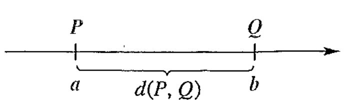


Figure 12

***That is, the distance between two points on a number line is the absolute value of the difference between their coordinates in either order. See* Figure 12.**

EXAMPLE 10 Finding the Distance between Two Points

Find the distance between -5 and 8.

SOLUTION Use the first formula above, with $a = -5$ and $b = 8$.

$$|b - a| = |8 - (-5)| = |8 + 5| = |13| = 13$$

Alternatively, for $a = 8$ and $b = -5$, we obtain the same result.

$$|b - a| = |(-5) - 8| = |-13| = 13$$

Now Try Exercise 105.